

WIENER TAUBERIAN THEOREM FOR HYPERGEOMETRIC TRANSFORMS

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ABSTRACT. We prove a genuine analogue of Wiener Tauberian theorem for hypergeometric transforms. As an application we prove analogue of Furstenberg theorem on Harmonic functions.

1. INTRODUCTION

A famous theorem of Norbert Wiener states that for a function $f \in L^1(\mathbb{R})$, span of translates $f(x - a)$ with complex coefficients is dense in $L^1(\mathbb{R})$ if and only if the Fourier transform \hat{f} is nonvanishing on \mathbb{R} . This theorem has been extended to abelian groups. The hypothesis (in the abelian case) is on a Haar integrable function which has nonvanishing Fourier transform on all unitary characters. However, Ehrenpreis and Mautner (in [5]) has observed that Wiener Tauberian theorem fails even for the commutative Banach algebra of integrable radial functions on $\mathrm{SL}(2, \mathbb{R})$. A modified version of the theorem was established in [5, Theorem 6] for radial functions in $L^1(\mathrm{SL}(2, \mathbb{R}))$. In their theorem they prove that if a function f satisfies “not-to-rapidly decay” condition and nonvanishing condition on some *extended strip*, etc., then the ideal generated by f is dense in $L^1(\mathrm{SL}(2, \mathbb{R})/\mathrm{SO}(2))$. This has been extended to all rank one semisimple Lie groups in the K -biinvariant setting (see [1], [18]) with the extended strip condition. The same theorem has been extended for hypergeometric transforms (in [12]). Further references in this literature are [17], [20], [21], [13], [14], [15]. In ([2, 3]), a genuine analogue of Wiener Tauberian theorem is proved for $\mathrm{SL}(2, \mathbb{R})$ in the K -biinvariant setting without the extended strip condition. Following their method we have extended this result to all real rank one semisimple Lie groups in the K -biinvariant settings ([16]). In this paper we prove Wiener Tauberian theorem for hypergeometric transforms in the exact strip. Let $\alpha \geq \beta \geq -\frac{1}{2}$, $\alpha \neq -\frac{1}{2}$, $\rho = \alpha + \beta + 1$, $S_1 = \{\lambda \in \mathbb{C} \mid |\Im \lambda| \leq \rho\}$ and

$$\Delta_{\alpha, \beta}(t) = (2|\sinh t|)^{2\alpha+1} (2 \cosh t)^{2\beta+1}, \quad t \in \mathbb{R}.$$

Let $L^1(\mathbb{R}, \Delta_{\alpha, \beta})_e$ be the collection of even functions f such that $\|f\|_1 := \int_{\mathbb{R}} |f(t)| \Delta_{\alpha, \beta}(t) dt < \infty$. Also let $L^1_0(\mathbb{R}, \Delta_{\alpha, \beta})_e$ be the collection of functions $f \in L^1(\mathbb{R}, \Delta_{\alpha, \beta})_e$ such that

$$\int_{\mathbb{R}} f(t) \Delta_{\alpha, \beta}(t) dt = 0.$$

For $f \in L^1(\mathbb{R}, \Delta_{\alpha, \beta})_e$, $\hat{f} = \hat{f}^{(\alpha, \beta)}$ denotes the Fourier-Jacobi transform of f (see preliminaries for the definition).

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For any function F on S_1 , we define

$$\delta_\infty^+(F) = -\limsup_{t \rightarrow \infty} e^{-\frac{\pi}{2\rho}t} \log |F(t)|, \quad \text{and} \quad \delta_{i\rho}(F) = \limsup_{t \rightarrow \rho^-} (\rho - t) \log |F(it)|.$$

Our theorem states that,

Theorem 1.1. *Let $\{f_\nu \mid \nu \in \Lambda\}$ be a collection of functions in $L^1(\mathbb{R}, \Delta_{\alpha,\beta})_e$ and I be the smallest closed ideal in $L^1(\mathbb{R}, \Delta_{\alpha,\beta})_e$ containing $\{f_\nu \mid \nu \in \Lambda\}$.*

- (1) *Suppose that element of $\{\widehat{f}_\nu \mid \nu \in \Lambda\}$ has no common zero in S_1 and $\inf_{\nu \in \Lambda} \delta_\infty^+(\widehat{f}_\nu) = 0$. Then $I = L^1(\mathbb{R}, \Delta_{\alpha,\beta})_e$.*
- (2) *Suppose that $\{\pm i\rho\}$ is the only common zero of $\{\widehat{f}_\nu \mid \nu \in \Lambda\}$ in S_1 and $\inf_{\nu \in \Lambda} \delta_\infty^+(\widehat{f}_\nu) = \inf_{\nu \in \Lambda} \delta_{i\rho}(\widehat{f}_\nu) = 0$. Then $I = L_0^1(\mathbb{R}, \Delta_{\alpha,\beta})_e$.*

Most of the part of the proof of this theorem similar to our earlier paper ([16]). Therefore we state such results without any proof. The proofs will follow as in [16].

As an application of this theorem we prove Frustenburg type theorem on Harmonic functions, following [3].

2. PRELIMINARIES

Most of our notation related to the hypergeometric functions is standard and can be found for example in [10]. We shall follow the standard practice of using the letter C for constants, whose value may change from one line to another. Everywhere in this article the symbol $f_1 \asymp f_2$ for two positive expressions f_1 and f_2 means that there are positive constants C_1, C_2 such that $C_1 f_1 \leq f_2 \leq C_2 f_1$. For a complex number z , we will use $\Re z$ and $\Im z$ to denote respectively the real and imaginary parts of z .

A Jacobi function $\phi_\lambda^{(\alpha,\beta)}(\alpha, \beta, \lambda \in \mathbb{C}, \alpha \neq -1, -2, \dots)$ is defined as the even C^∞ function on \mathbb{R} such that $\phi_\lambda^{(\alpha,\beta)}(0) = 1$ and it satisfies the following differential equation

$$\left(\frac{d^2}{dt^2} + ((2\alpha + 1) \coth t + (2\beta + 1) \tanh t) \frac{d}{dt} + \lambda^2 + (\alpha + \beta + 1)^2 \right) \phi_\lambda^{(\alpha,\beta)}(t) = 0. \quad (2.1)$$

In this paper we shall assume that $\alpha \geq \beta > -\frac{1}{2}$ and $\alpha \neq -\frac{1}{2}$. This Jacobi function can be written as hypergeometric function:

$$\phi_\lambda^{(\alpha,\beta)}(t) = {}_2F_1 \left(\frac{\alpha + \beta + 1 - i\lambda}{2}, \frac{\alpha + \beta + 1 + i\lambda}{2}; \alpha + 1; -\sinh^2 t \right). \quad (2.2)$$

The hypergeometric function has the following integral representation for $\Re c > \Re b > 0$,

$${}_2F_1(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 s^{b-1} (1-s)^{c-b-1} (1-sz)^{-a} ds, \quad z \in \mathbb{C} \setminus [1, \infty). \quad (2.3)$$

Let

$$L_{\alpha,\beta} = \frac{d^2}{dt^2} + ((2\alpha + 1) \coth t + (2\beta + 1) \tanh t) \frac{d}{dt}.$$

Then rewriting (2.1) we get that $\phi_\lambda^{(\alpha,\beta)}$ is the unique even C^∞ function on \mathbb{R} such that $\phi_\lambda^{(\alpha,\beta)}(0) = 1$ and

$$(L_{\alpha,\beta} + \lambda^2 + \rho^2) \phi_\lambda^{(\alpha,\beta)} = 0. \quad (2.4)$$

Let $T^{(\alpha,\beta)}$ be the differential-difference operator defined by

$$T^{(\alpha,\beta)}f(t) = f'(t) + ((2\alpha + 1)\coth t + (2\beta + 1)\tanh t) \frac{f(t) - f(-t)}{2} - \rho f(-t), t \in \mathbb{R}.$$

The Heckman-Opdam hypergeometric functions $G_\lambda^{(\alpha,\beta)}$ on \mathbb{R} are normalised eigenfunctions:

$$T^{(\alpha,\beta)}G_\lambda^{(\alpha,\beta)} = i\lambda G_\lambda^{(\alpha,\beta)}.$$

The functions $G_\lambda^{(\alpha,\beta)}$ are related to the Jacobi functions by

$$G_\lambda^{(\alpha,\beta)}(x) = \phi_\lambda^{(\alpha,\beta)}(x) + \frac{\rho + i\lambda}{4(\alpha + 1)} \sinh 2x \phi_\lambda^{(\alpha+1,\beta+1)}(x). \quad (2.5)$$

Then we have,

$$\phi_\lambda^{(\alpha,\beta)}(x) = \frac{G_\lambda^{(\alpha,\beta)}(x) + G_\lambda^{(\alpha,\beta)}(-x)}{2}.$$

For $\lambda \neq -i, -2i, \dots$, there is another solution $\Phi_\lambda^{(\alpha,\beta)}$ of (2.4) on $(0, \infty)$ is given by

$$\Phi_\lambda^{(\alpha,\beta)}(t) = (2 \cosh t)^{i\lambda-\rho} {}_2F_1\left(\frac{\rho - i\lambda}{2}, \frac{\alpha - \beta + 1 - i\lambda}{2}; 1 - i\lambda; \cosh^{-2} t\right) \quad (2.6)$$

$$= (2 \sinh t)^{i\lambda-\rho} {}_2F_1\left(\frac{\rho - i\lambda}{2}, \frac{-\alpha + \beta + 1 - i\lambda}{2}; 1 - i\lambda; -\sinh^{-2} t\right) \quad (2.7)$$

This solution has singularity at $t = 0$. For $t \rightarrow \infty$, it satisfies

$$\Phi_\lambda^{(\alpha,\beta)}(t) = e^{(i\lambda-\rho)t}(1 + O(1)). \quad (2.8)$$

For $\lambda \in \mathbb{C} \setminus i\mathbb{Z}$, $\Phi_\lambda^{(\alpha,\beta)}$ and $\Phi_{-\lambda}^{(\alpha,\beta)}$ are two linearly independent solutions of (2.4). So $\phi_\lambda^{(\alpha,\beta)}$ is a linear combination of both $\Phi_\lambda^{(\alpha,\beta)}$ and $\Phi_{-\lambda}^{(\alpha,\beta)}$. We have

$$\phi_\lambda^{(\alpha,\beta)} = c(\lambda)\Phi_\lambda^{(\alpha,\beta)} + c(-\lambda)\Phi_{-\lambda}^{(\alpha,\beta)}$$

where $c(\lambda)$ is the Harish-Chandra c -function given by

$$c(\lambda) = c_{(\alpha,\beta)}(\lambda) = \frac{2^{\rho-i\lambda}\Gamma(\alpha+1)\Gamma(i\lambda)}{\Gamma(\frac{\rho+i\lambda}{2})\Gamma(\frac{i\lambda+\alpha-\beta+1}{2})}.$$

It is normalized such that $c(-i\rho) = 1$. Hence, for $\Im \lambda < 0$ and as $t \rightarrow \infty$,

$$\phi_\lambda^{(\alpha,\beta)}(t) = c(\lambda)e^{(i\lambda-\rho)t}(1 + O(1)). \quad (2.9)$$

We let $\Delta_{\alpha,\beta}(t) = (2|\sinh t|)^{2\alpha+1}(2 \cosh t)^{2\beta+1}, t \in \mathbb{R}$. The Fourier-Jacobi transform of a suitable even function f on \mathbb{R} is defined by

$$\widehat{f}^{(\alpha,\beta)}(\lambda) = \int_{\mathbb{R}} f(t)\phi_\lambda^{(\alpha,\beta)}(t)\Delta_{(\alpha,\beta)}(t) dt = 2 \int_0^\infty f(t)\phi_\lambda^{(\alpha,\beta)}(t)\Delta_{(\alpha,\beta)}(t) dt$$

for all complex numbers λ , for which the right hand side is well-defined. We point out that this definition coincides exactly with the group Fourier transform when (α, β) arises from geometric cases.

The Fourier-Jacobi transform of an even complex Borel measure μ is defined by

$$\widehat{\mu}^{(\alpha,\beta)}(\lambda) = \int_{\mathbb{R}} \phi_\lambda^{(\alpha,\beta)}(t) d\mu(t).$$

for $\lambda \in S_1$.

For $f \in L^1(\mathbb{R}, \Delta_{\alpha,\beta})_e$, we have $\lim_{|\lambda| \rightarrow \infty} \widehat{f}^{(\alpha,\beta)}(\lambda) = 0$ and $\lim_{|\lambda| \rightarrow \infty} \widehat{\mu}^{(\alpha,\beta)}(\lambda) = \mu(\{0\})$.

Also we have the following inversion formula for suitable even function f on \mathbb{R} :

$$f(t) = \frac{1}{4\pi} \int_0^\infty \widehat{f}^{(\alpha, \beta)}(\lambda) \phi_\lambda^{(\alpha, \beta)}(t) |c_{(\alpha, \beta)}(\lambda)|^{-2} d\lambda.$$

The translation of a suitable even function f on \mathbb{R} is given by (for all $s, t \in \mathbb{R}$),

$$\tau_s^{(\alpha, \beta)} f(t) = \int_0^1 \int_0^\pi f \left(\cosh^{-1} \left| \cosh s \cosh t + r e^{i\psi} \sinh s \sinh t \right| \right) dm_{\alpha, \beta}(r, \psi)$$

where the measure $dm_{\alpha, \beta}(r, \psi)$ is given by

$$dm_{\alpha, \beta}(r, \psi) = \frac{2\Gamma(\alpha + 1)}{\Gamma(\frac{1}{2})\Gamma(\alpha - \beta)\Gamma(\beta + \frac{1}{2})} (1 - r^2)^{\alpha - \beta - 1} (r \sin \psi)^{2\beta} r dr d\psi$$

for $\alpha > \beta > -\frac{1}{2}$.

For $\alpha = \beta > -\frac{1}{2}$ the measure degenerates into

$$dm_{\alpha, \alpha}(r, \psi) = \frac{\Gamma(\alpha + 1)}{\Gamma(\frac{1}{2})\Gamma(\alpha + \frac{1}{2})} (\sin \psi)^{2\alpha} d\psi d\delta_0(r)$$

and for $\alpha > \beta = -\frac{1}{2}$ into

$$dm_{\alpha, -\frac{1}{2}}(r, \psi) = \frac{2\Gamma(\alpha + 1)}{\Gamma(\frac{1}{2})\Gamma(\alpha + \frac{1}{2})} (1 - r^2)^{\alpha - \frac{1}{2}} dr \frac{1}{2} d(\delta_0 + \delta_\pi)(\psi).$$

Then it easy to check that

- (1) $\tau_s^{(\alpha, \beta)} f(t) = \tau_t^{(\alpha, \beta)} f(s)$
- (2) $\tau_0^{(\alpha, \beta)} f = f$
- (3) $\tau_{-s}^{(\alpha, \beta)} f(t) = \tau_s^{(\alpha, \beta)} f(-t)$
- (4) $\tau_s^{(\alpha, \beta)} \tau_t^{(\alpha, \beta)} = \tau_t^{(\alpha, \beta)} \tau_s^{(\alpha, \beta)}$.
- (5) $\tau_s^{(\alpha, \beta)} \phi_\lambda^{(\alpha, \beta)}(t) = \phi_\lambda^{(\alpha, \beta)}(s) \phi_\lambda^{(\alpha, \beta)}(t)$.

- (6) For suitable even function f on \mathbb{R} , we have $\widehat{\tau_s^{(\alpha, \beta)} f}(\lambda) = \phi_\lambda^{(\alpha, \beta)}(s) \widehat{f}^{(\alpha, \beta)}(\lambda)$.

For suitable even functions f and g the convolution on \mathbb{R} is defined by

$$f *_{(\alpha, \beta)} g(t) = \int_{\mathbb{R}} \left(\tau_s^{(\alpha, \beta)} f \right) (t) g(s) \Delta_{(\alpha, \beta)}(s) ds. \quad (2.10)$$

Also the convolution of a suitable even function f and an even complex measure μ is defined by

$$f *_{(\alpha, \beta)} \mu(t) = \int_{\mathbb{R}} \left(\tau_s^{(\alpha, \beta)} f \right) (t) d\mu(s). \quad (2.11)$$

It is well known that

$$\widehat{f *_{(\alpha, \beta)} g}(\lambda) = \widehat{f}(\lambda) \widehat{g}(\lambda)$$

and

$$\|f *_{(\alpha, \beta)} g\|_1 \leq \|f\|_1 \|g\|_1.$$

3. THE FUNCTIONS b_λ

Let $\mathbb{C}_+ = \{z \in \mathbb{C} \mid \Im z > 0\}$ be the open upper half plane in \mathbb{C} . We fix $\alpha \geq \beta \geq -\frac{1}{2}, \alpha \neq -\frac{1}{2}$. To make expressions simpler, we shall omit indices (α, β) from $\Phi_\lambda^{(\alpha, \beta)}, \phi_\lambda^{(\alpha, \beta)}, \Delta_{(\alpha, \beta)}, c_{(\alpha, \beta)}(\lambda), \dots$ etc. and write simply them as $\Phi_\lambda, \phi_\lambda, \Delta, c(\lambda), \dots$ etc. respectively.

For $\lambda \in \mathbb{C}_+$, we define

$$b_\lambda(t) := \frac{i}{4\lambda c(-\lambda)} \Phi_\lambda(t), t > 0 \quad (3.1)$$

where c is the Harish-Chandra c -function. We extend b_λ evenly on $\mathbb{R} \setminus \{0\}$. The function b_λ satisfies the following properties.

- (1) There is a positive constant C and a natural number N such that for all $t \in (0, 1/2]$,

$$|b_\lambda(t)| \leq \begin{cases} C(1 + |\lambda|)^N t^{-2\alpha}, & \text{if } \alpha \neq 0 \\ C \log \frac{1}{t} & \text{if } \alpha = 0. \end{cases}$$

- (2) There is a positive constant C and a natural number M such that for all $t \in [1/2, \infty]$,

$$|b_\lambda(at)| \leq C(1 + |\lambda|)^M e^{-(\Im \lambda + \rho)t}.$$

- (3) b_λ can be written as a sum of L^1 and $L^p(p < 2)$ functions.

- (4) $b_\lambda \in L^1(\mathbb{R}, \Delta)_e$ if and only if $\Im \lambda > \rho$ and $\|b_\lambda\|_1 \leq C \frac{(1+|\lambda|)^K}{\Im \lambda - \rho}$ for some $K > 0$. Also, $\|b_\lambda\|_1 \rightarrow 0$ if $\lambda \rightarrow \infty$ along the positive imaginary axis.

- (5) For $\lambda \in \mathbb{C}_+$, $\widehat{b}_\lambda(\xi) = \frac{1}{\xi^2 - \lambda^2}, \xi \in \mathbb{R}$.

- (6) $\text{Span}\{b_\lambda \mid \Im \lambda > \rho\}$ is dense in $L^1(\mathbb{R}, \Delta)_e$.

Except (5), others can be proved as in [16]. So we present the proof of (5) (cf. [22, p. 128]).

Lemma 3.1. *Let $\lambda \in \mathbb{C}_+$. Then $\widehat{b}_\lambda(\xi) = \frac{1}{\xi^2 - \lambda^2}$ for all $\xi \in \mathbb{R}$.*

Proof. For two smooth functions f and g on $(0, \infty)$, we define

$$[f, g](t) = \Delta(t) [f(t)g'(t) - f'(t)g(t)], \quad t > 0.$$

An easy calculation shows that $[f, g]'(t) = (Lf \cdot g - f \cdot Lg)(t)\Delta(t)$. Therefore, for any $b > a > 0$, we have

$$\int_a^b (Lf \cdot g - f \cdot Lg)(t)\Delta(t) dt = [f, g](b) - [f, g](a). \quad (3.2)$$

If $f = \phi_\lambda$ and $g = \Phi_\lambda$, then the left-hand side of the above equation is zero for all $b > a > 0$, so that $[\phi_\lambda, \Phi_\lambda]$ is a (finite) constant on $(0, \infty)$, and hence

$$[\phi_\lambda, \Phi_\lambda](\cdot) = \lim_{t \rightarrow \infty} [\phi_\lambda, \Phi_\lambda] = - \lim_{t \rightarrow \infty} \Delta(t) (\Phi_\lambda(t))^2 \left(\frac{\phi_\lambda}{\Phi_\lambda} \right)'(t) = - \lim_{t \rightarrow \infty} e^{2i\lambda t} \left(\frac{\phi_\lambda}{\Phi_\lambda} \right)'(t) \quad (3.3)$$

by the asymptotic behaviors of Δ and Φ_λ at ∞ . Again, by the asymptotic behaviors of ϕ_λ and Φ_λ at ∞ , we have

$$\lim_{t \rightarrow \infty} \frac{\frac{\phi_\lambda}{\Phi_\lambda}(t)}{e^{-2i\lambda t}} = c(-\lambda).$$

Since, by (3.3),

$$\lim_{t \rightarrow \infty} \frac{\left(\frac{\phi_\lambda}{\Phi_\lambda} \right)'(t)}{-2i\lambda e^{-2i\lambda t}}$$

exists, we can apply L'Hospital's rule to conclude that

$$\lim_{t \rightarrow \infty} \frac{\left(\frac{\phi_\lambda}{\Phi_\lambda}\right)'(t)}{-2i\lambda e^{-2i\lambda t}} = c(-\lambda).$$

Hence we get $[\phi_\lambda, \Phi_\lambda](\cdot) = 2i\lambda c(-\lambda)$.

Now, if f is an even smooth function on \mathbb{R} with $f(0) = 0$, we claim that

$$\lim_{t \rightarrow 0^+} [f, \Phi_\lambda](t) = 0.$$

To prove the claim, first note that we can assume f to be compactly supported. Then with this f and $g = \Phi_\lambda$, the equation 3.2 (for large b and $a = t \rightarrow 0^+$) implies that $\lim_{t \rightarrow 0^+} [f, \Phi_\lambda](t)$ exists. Also we have

$$\lim_{t \rightarrow 0^+} [f, \Phi_\lambda](t) = - \lim_{t \rightarrow 0^+} \Delta(t) (\Phi_\lambda(t))^2 \left(\frac{f}{\Phi_\lambda}\right)'(t) = \begin{cases} - \lim_{t \rightarrow 0^+} 2^{2\rho} C t^{-2\alpha+1} \left(\frac{f}{\Phi_\lambda}\right)'(t), & \text{if } \alpha \neq 0 \\ - \lim_{t \rightarrow 0^+} 2^{2\rho} C t \left(\log \frac{1}{t}\right)^2 \left(\frac{f}{\Phi_\lambda}\right)'(t), & \text{if } \alpha = 0 \end{cases}$$

since $\Delta(t) \asymp 2^{2\rho} t^{2\alpha+1}$ and

$$\Phi_\lambda(t) \asymp \begin{cases} t^{-2\alpha} & \text{if } \alpha \neq 0 \\ \log \frac{1}{t} & \text{if } \alpha = 0 \end{cases} \quad (3.4)$$

as $t \rightarrow 0^+$. Therefore, by an application of L'Hospital's rule, the claim follows. But, if $f(0) \neq 0$, writing

$$[f, \Phi_\lambda] = [f - f(0)\phi_\lambda, \Phi_\lambda] + f(0)[\phi_\lambda, \Phi_\lambda],$$

we can conclude that

$$\lim_{t \rightarrow 0^+} [f, \Phi_\lambda](t) = 2i\lambda c(-\lambda)f(0). \quad (3.5)$$

Now fix a real number ξ . Putting $f = \phi_\xi$ and $g = \Phi_\lambda$ in 3.2, it follows that

$$\int_a^b \Phi_\lambda(t) \phi_\xi(t) \Delta(t) dt = \frac{1}{\lambda^2 - \xi^2} ([\phi_\xi, \Phi_\lambda](b) - [\phi_\xi, \Phi_\lambda](a)).$$

Taking limit as $a \rightarrow 0^+$, we get, by 3.5,

$$\int_0^b \Phi_\lambda(t) \phi_\xi(t) \Delta(t) dt = \frac{1}{\lambda^2 - \xi^2} ([\phi_\xi, \Phi_\lambda](b) - 2i\lambda c(-\lambda)).$$

Therefore, to complete the proof it is enough to show that $[\phi_\xi, \Phi_\lambda](b) \rightarrow 0$ as $b \rightarrow \infty$. First note that the existence of (finite) limit is confirmed by the above equation itself. As in 3.3, we can write

$$\lim_{b \rightarrow \infty} [\phi_\xi, \Phi_\lambda](b) = \lim_{b \rightarrow \infty} e^{2i\lambda b} \left(\frac{\phi_\xi}{\Phi_\lambda}\right)'(b).$$

By the asymptotic behavior of ϕ_ξ and Φ_λ ,

$$\lim_{b \rightarrow \infty} \frac{\frac{\phi_\xi}{\Phi_\lambda}(b)}{e^{-2i\lambda b}} = 0.$$

Therefore, by L'Hospital rule,

$$\lim_{b \rightarrow \infty} \frac{\left(\frac{\phi_\xi}{\Phi_\lambda}\right)'(b)}{-2i\lambda e^{-2i\lambda b}} = 0,$$

and hence $\lim_{b \rightarrow \infty} [\phi_\xi, \Phi_\lambda](b) = 0$ as required to prove. \square

4. THE FUNCTIONS $T_\lambda f$

Let $f \in L^1(\mathbb{R}, \Delta)_e$. For each λ , with $0 < \Im \lambda < \rho$, we define

$$T_\lambda f := \widehat{f}(\lambda) b_\lambda - f * b_\lambda. \quad (4.1)$$

Since b_λ can be written as a sum of L^1 and L^p ($p < 2$) function, $T_\lambda f$ is well-defined; in fact it also has the same form i.e. can be written as a sum of L^1 and L^p function. In particular its spherical transform is a continuous function on \mathbb{R} . As an easy consequence of Lemma 3.1 we get, for $0 < \Im \lambda < \rho$ and $f \in L^1(\mathbb{R}, \Delta)_e$,

$$\widehat{T_\lambda f}(\xi) = \frac{\widehat{f}(\lambda) - \widehat{f}(\xi)}{\xi^2 - \lambda^2}, \quad \text{for all } \xi \in \mathbb{R}.$$

Lemma 4.1. *Let $\lambda \in \mathbb{C}_+$. Then*

$$\tau_s b_\lambda(t) = \begin{cases} b_\lambda(t) \phi_\lambda(s) & \text{if } t > s \geq 0, \\ b_\lambda(s) \phi_\lambda(t) & \text{if } s > t \geq 0. \end{cases}$$

Proof. First we note that if $t \neq s$, $\cosh^{-1}(|\cosh s \cosh t + r e^{i\psi} \sinh s \sinh t|)$ is non zero, whatever the value of $r \in [0, 1]$ and $\psi \in [0, \pi]$ be. Therefore $\tau_s b_\lambda(t)$ is well-defined whenever $t \neq s$. Since $\tau_s b_\lambda(t) = \tau_t b_\lambda(s)$, it is enough to prove the second case. Fix $s > 0$. Since b_λ is an smooth eigenfunction of L on $(0, \infty)$ with eigen value $-(\lambda^2 + \rho^2)$, $\tau_s b_\lambda$ is an smooth eigenfunction of L on $(0, s)$ with eigenvalue $-(\lambda^2 + \rho^2)$ which is regular at 0. Therefore

$$\tau_s b_\lambda(t) = C \phi_\lambda(t) \quad \text{for all } 0 \leq t < s,$$

for some constant C . Putting $t = 0$ in the above equation we get $C = b_\lambda(s)$. Hence the proof. \square

Using Lemma 4.1 $T_\lambda f$, $0 < \Im \lambda < \rho$ can be written as,

$$T_\lambda f(t) = b_\lambda(t) \int_t^\infty f(s) \phi_\lambda(s) \Delta(s) ds - \phi_\lambda(t) \int_t^\infty f(s) b_\lambda(s) \Delta(s) ds, t > 0.$$

Using this expression of $T_\lambda f$, we can prove the following lemma (see Lemma 4.4, Remark 4.5 [16]).

Lemma 4.2. *Let $0 < \Im \lambda < \rho$ and $f \in L^1(\mathbb{R}, \Delta)_e$. Also assume that $\lambda \notin B_{\rho/2}(0)$. Then $T_\lambda f \in L^1(\mathbb{R}, \Delta)_e$ and its L^1 norm satisfies $\|T_\lambda f\|_1 \leq C \|f\|_1 (1 + |\lambda|)^L d(\lambda, \partial S_1)^{-1}$, for some non-negative integer L , where $d(\lambda, \partial S_1)$ denotes the Euclidean distance of λ from the boundary ∂S_1 of the strip S_1 .*

5. RESOLVENT TRANSFORM

Let δ be the Dirac delta distribution at 0. Let $L_\delta^1(\mathbb{R}, \Delta)_e$ be the unital Banach algebra generated by $L^1(\mathbb{R}, \Delta)_e$ and $\{\delta\}$. Its maximal ideal space is one point compactification $S_1 \cup \{\infty\}$ of S_1 , i.e., more precisely, the maximal ideal space is $\{L_z : z \in S_1 \cup \{\infty\}\}$, where L_z is the complex homomorphism on $L_\delta^1(\mathbb{R}, \Delta)_e$ defined by $L_z(f) = \widehat{f}(z)$. For a (closed) ideal J of $L_\delta^1(\mathbb{R}, \Delta)_e$, the hull $Z(J)$ is defined to be the set of common zeros (in $S_1 \cup \{\infty\}$) of the Jacobi-Fourier transforms of elements in J . For the rest of the section I always stands for a (closed) ideal of $L^1(\mathbb{R}, \Delta)_e$ (and hence an ideal in $L_\delta^1(\mathbb{R}, \Delta)_e$ too) such that the hull $Z = Z(I)$ is $\{\infty\}$ or $\{\infty, \pm i\rho\}$. Since Z is the set of common zeros of Jacobi-Fourier transforms of the elements in I , it follows that the maximal ideal space of the quotient algebra $L_\delta^1(\mathbb{R}, \Delta)_e/I$ is Z i.e. it

consists of the complex homomorphisms $\tilde{L}_z : z \in Z$, where $\tilde{L}_z(f + I) = \hat{f}(z)$. So, by the Banach algebra theory, an element $f + I$ is invertible in $L_\delta^1(\mathbb{R}, \Delta)_e/I$ iff $\hat{f}(z) \neq 0$ for all $z \in Z$.

Let λ_0 be a fixed complex number with $\Im \lambda_0 > \rho$, so that b_{λ_0} is in L^1 . Therefore, for $\lambda \in \mathbb{C} \setminus Z$, the function $\hat{\delta} - (\lambda^2 - \lambda_0^2)\hat{b}_{\lambda_0}$ does not vanish at any points of Z , and hence $\delta - (\lambda^2 - \lambda_0^2)b_{\lambda_0} + I$ is invertible in the quotient algebra $L_\delta^1(\mathbb{R}, \Delta)_e/I$. We put

$$B_\lambda = (\delta - (\lambda^2 - \lambda_0^2)b_{\lambda_0} + I)^{-1} * (b_{\lambda_0} + I), \quad \lambda \in \mathbb{C} \setminus Z \quad (5.1)$$

which is, in fact, an element of $L^1(\mathbb{R}, \Delta)_e/I$. Now, let $g \in L^\infty(\mathbb{R}, \Delta)_e$ annihilates I , so that we may consider g as a bounded linear functional on $L^1(\mathbb{R}, \Delta)_e/I$. We define the resolvent transform $\mathcal{R}[g]$ of g by

$$\mathcal{R}[g](\lambda) = \langle B_\lambda, g \rangle \quad (5.2)$$

From (5.1) it is easy to see that $\lambda \mapsto B_\lambda$ is a Banach space valued even holomorphic function on $\mathbb{C} \setminus Z$. It follows that $\mathcal{R}[g]$ is an even holomorphic function on $\mathbb{C} \setminus Z$.

The resolvent transform $\mathcal{R}[g]$ has the following properties. The proof of this lemma is same as that of Lemma 5.1 in [16]. But we present the proof here since the lemma is the core of the proof of the Wiener Tauberian theorem.

Lemma 5.1. *Assume $g \in L^\infty(\mathbb{R}, \Delta)_e$ annihilates I , and fix a function $f \in I$. Let $Z(\hat{f}) := \{z \in S_1 : \hat{f}(z) = 0\}$.*

(a) $\mathcal{R}[g](\lambda)$ is an even holomorphic function on $\mathbb{C} \setminus Z$. It is given by the following formula :

$$\mathcal{R}[g](\lambda) = \begin{cases} \langle b_\lambda, g \rangle, & \Im \lambda > \rho, \\ \frac{\langle T_\lambda f, g \rangle}{\hat{f}(\lambda)}, & 0 < \Im \lambda < \rho, \lambda \notin Z(\hat{f}). \end{cases}$$

(b) For $|\Im \lambda| > \rho$, $|\mathcal{R}[g](\lambda)| \leq C \|g\|_\infty \frac{(1+|\lambda|)^K}{d(\lambda, \partial S_1)}$,

(c) For $|\Im \lambda| < \rho$, $\left| \hat{f}(\lambda) \mathcal{R}[g](\lambda) \right| \leq C \|f\|_1 \|g\|_\infty \frac{(1+|\lambda|)^L}{d(\lambda, \partial S_1)}$, where the constant C is independent of $f \in I$.

Proof. (a) Let $\Im \lambda > \rho$. Then b_λ is in L^1 and $\hat{b}_\lambda(z) = \frac{1}{z^2 - \lambda^2}$, $z \in S_1$. We observe that for $z \in S_1$,

$$\frac{1}{\hat{b}_{\lambda_0}(z)} - \frac{1}{\hat{b}_\lambda(z)} = \lambda^2 - \lambda_0^2$$

which is equivalent to saying that

$$\left(1 - (\lambda^2 - \lambda_0^2)\hat{b}_{\lambda_0}(z)\right)\hat{b}_\lambda(z) = \hat{b}_{\lambda_0}(z), \quad z \in S_1.$$

Apply the inverse spherical transform and mod out I to get

$$(\delta - (\lambda^2 - \lambda_0^2)b_{\lambda_0} + I) * (b_\lambda + I) = b_{\lambda_0} + I,$$

Since $(\delta - (\lambda^2 - \lambda_0^2)b_{\lambda_0} + I)$ is invertible in $L_\delta^1(\mathbb{R})_e/I$, comparing the above equation with 5.1 we get $B_\lambda = b_\lambda + I$. Therefore, by the definition of $\mathcal{R}[g](\lambda)$, $\mathcal{R}[g](\lambda) = \langle b_\lambda, g \rangle$.

Next we assume that $0 < \Im \lambda < \rho$, $\lambda \notin Z(\hat{f})$. So, $T_\lambda f$ is in L^1 and $\widehat{T_\lambda f}(z) = \frac{\hat{f}(\lambda) - \hat{f}(z)}{z^2 - \lambda^2}$, $z \in S_1$. A small calculation shows that

$$\left(1 - (\lambda^2 - \lambda_0^2)\hat{b}_{\lambda_0}(z)\right) \frac{\widehat{T_\lambda f}(z)}{\hat{f}(\lambda)} = \hat{b}_{\lambda_0}(z) - \frac{\hat{f}(z)\hat{b}_{\lambda_0}(z)}{\hat{f}(\lambda)}, \quad z \in S_1.$$

Again, apply inverse spherical transform and mod out I to get

$$(\delta - (\lambda^2 - \lambda_0^2)b_{\lambda_0} + I) * \left(\frac{T_{\lambda}f}{\widehat{f}(\lambda)} + I \right) = b_{\lambda_0} + I.$$

Therefore $B_{\lambda} = \frac{T_{\lambda}f}{\widehat{f}(\lambda)} + I$ which gives the desired formula for $\mathcal{R}[g](\lambda)$ in this case.

- (b) It follows from the estimate of $\|b_{\lambda}\|_1$ and the fact that $\mathcal{R}[g](\lambda)$ is even.
- (c) From Lemma 4.2 it follows that

$$\left| \widehat{f}(\lambda)\mathcal{R}[g](\lambda) \right| \leq C\|f\|_1\|g\|_{\infty} \frac{(1+|\lambda|)^L}{d(\lambda, \partial S_1)}$$

for $0 < \Im \lambda < \rho/2, \lambda \notin B_{\rho/2}(0)$, where C is independent of $f \in I$. Since $\widehat{f}(\lambda)\mathcal{R}[g](\lambda)$ is an even continuous function on S_1 , the same estimate is true for $|\Im \lambda| < \rho, \lambda \notin B_{\rho/2}(0)$. From (5.2) it follows that $\mathcal{R}[g](\lambda)$ is bounded on $B_{\rho/2}(0)$, with bound independent of f . Therefore on $B_{\rho/2}(0)$

$$\left| \widehat{f}(\lambda)\mathcal{R}[g](\lambda) \right| \leq C\|f\|_1$$

where C is independent of f . Hence the proof follows. \square

6. SOME RESULTS FROM COMPLEX ANALYSIS

In this section we state some results from complex analysis. The proof of them involves the log-log theorem, the Paley-Wiener theorem, Ahlfors distortion theorem, and the Phragman-Lindel'of principle ([8], [4]).

For any function F on \mathbb{R} , we let

$$\delta_{\infty}^{+}(F) = -\limsup_{t \rightarrow \infty} e^{-\frac{\pi}{2\rho}t} \log |F(t)|, \quad \delta_{\infty}^{-}(F) = -\limsup_{t \rightarrow \infty} e^{-\frac{\pi}{2\rho}t} \log |F(-t)|$$

and

$$\delta_{i\rho}(F) = \limsup_{t \rightarrow \rho-} (\rho - t) \log |F(it)|, \quad \delta_{-i\rho}(F) = \limsup_{t \rightarrow (-\rho)+} (\rho + t) \log |F(it)|.$$

Proof of the following theorem is similar to [16, Theorem 6.3].

Theorem 6.1. *Let Ω be a collection of bounded holomorphic functions F on S_1^0 such that*

$$\inf_{F \in \Omega} \delta_{\infty}^{+}(F) = \inf_{F \in \Omega} \delta_{\infty}^{-}(F) = 0.$$

Suppose H is a holomorphic function on $\mathbb{C} \setminus \{\pm i\rho\}$ such that, for some non-negative integer N , it satisfies the following estimates :

$$\begin{aligned} |H(z)| &\leq \frac{(1+|z|)^N}{d(z, \partial S_1)}, \quad z \in \mathbb{C} \setminus S_1, \\ |F(z)H(z)| &\leq \frac{(1+|z|)^N}{d(z, \partial S_1)}, \quad z \in S_1^0, \text{ for all } F \in \Omega. \end{aligned}$$

Then H is dominated by a polynomial outside a bounded neighbourhood of $\{\pm i\rho\}$.

The following theorem follows from the proof of [4, Theorem 6.13]:

Theorem 6.2. *Let Ω be a collection of bounded holomorphic functions F on S_1^0 such that $|F(z)| \rightarrow 0$ as $|z| \rightarrow \infty$ (in S_1^0) and*

$$\inf_{F \in \Omega} \delta_{\pm i\rho}(F) = 0.$$

Suppose G is a holomorphic function on $\mathbb{C} \setminus Z$ (Z is a finite subset of ∂S_1) such that for some positive integer N it satisfies the following estimate :

$$|F(z)G(z)| \leq (d(z, \partial S_1))^{-N}, \quad z \in S_1^0, \text{ for all } F \in \Omega.$$

Then G has poles at $\pm i\rho$ of order atmost N .

Theorem 6.3. *Let Ω be a collection of bounded holomorphic functions F on S_1^0 such that $|F(z)| \rightarrow 0$ as $|z| \rightarrow \infty$ (in S_1^0) and*

$$\inf_{F \in \Omega} \delta_{\pm i\rho}(F) = 0.$$

Suppose H is a holomorphic function on $\mathbb{C} \setminus \{\pm i\rho\}$ satisfying the following estimate (for some positive integer N) :

$$|F(z)H(z)| \leq \frac{(1 + |z|)^N}{d(z, \partial S_1)}, \quad z \in S_1^0, \text{ for all } F \in \Omega.$$

Then G has at most simple poles at $\pm i\rho$.

Proof. We can assume that N is even. We define the holomorphic function G on $\mathbb{C} \setminus \{\pm i\rho\}$ by

$$G(z) = \frac{H(z)}{(z - i\rho)^{N/2}(z + i\rho)^{N/2}}.$$

Then clearly $G(z)$ satisfies

$$|F(z)G(z)| \leq \frac{1}{(d(z, \partial S_1))^{N/2+1}}, \quad z \in S_1^0, \text{ for all } F \in \Omega.$$

Hence the theorem follows by the previous theorem. \square

7. PROOF OF THE MAIN THEOREM

proof of Theorem 1.1: Proof of **(1)** is similar to “*proof of Theorem 1.2*” in Section 7 (in [16]).

(2) We can assume that the elements in I are of unit norm. Let $g \in L^\infty(\mathbb{R}, \Delta)_e$ annihilates the (closed) ideal I generated by $\{f_\nu \mid \nu \in \Lambda\}$. We must show that g annihilates $L_0^1(\mathbb{R}, \Delta)_e$. By Lemma 5.1, $\mathcal{R}[g]$ satisfies the following estimates

$$\begin{aligned} |\mathcal{R}[g](z)| &\leq C(1 + |z|)^N (d(z, \partial S_1))^{-1}, \quad z \in \mathbb{C} \setminus S_1, \\ |\widehat{f}_\nu(z)\mathcal{R}[g](z)| &\leq C(1 + |z|)^N (d(z, \partial S_1))^{-1}, \quad z \in S_1^0, \end{aligned}$$

for all $\nu \in \Lambda$, for some constant C . Therefore, by Theorem 6.3, it has at most simple poles at $\{\pm i\rho\}$. So we write

$$\mathcal{R}[g](z) = \frac{a}{z^2 + \rho^2} + h(z), \quad z \in \mathbb{C} \setminus \{\pm i\rho\}$$

for some constant a and even entire function h . Also, by Theorem 6.1 $\mathcal{R}[g]$ has at most polynomial growth at ∞ , and by (4) (in section 3), $\mathcal{R}[g](z) \rightarrow 0$ as $|z| \rightarrow \infty$ along the imaginary axis. Therefore the same properties are satisfied by the function h too, so that by Liouville's theorem $h = 0$, and hence

$$\mathcal{R}[g](z) = \frac{a}{z^2 + \rho^2}, \quad z \in \mathbb{C} \setminus \{\pm i\rho\}.$$

Let $m \in L^\infty(\mathbb{R}, \Delta)_e$ corresponds to the complex homomorphism $f \mapsto \hat{f}(i\rho)$ on $L^1(\mathbb{R}, \Delta)_e$ i.e. $\hat{f}(i\rho) = \langle f, m \rangle$ for all $f \in L^1(\mathbb{R}, \Delta)_e$. Then for z with $\Im z > 0$,

$$\mathcal{R}[g](z) = -a\hat{b}_z(i\rho) = -a\langle b_z, m \rangle.$$

Since $\{b_z : \Im z > 0\}$ is dense in $L^1(\mathbb{R}, \Delta)_e$, $g = -\bar{a}m$. Since m annihilates $L_0^1(\mathbb{R}, \Delta)_e$, so does g . \square

8. FURSTENBERG THEOREM

Let G be a noncompact connected semisimple Lie group with finite center and K be a maximal compact subgroup of G . Let μ be a K -invariant complex measure on G/K such that $\mu(G/K) = 1$. A bounded function f on G/K is said to be μ -harmonic if $f * \mu = f$ i.e.

$$\int_G f(gh) d\mu(h) = f(g), \quad \text{for all } g \in G.$$

If f is harmonic (i.e. $\int_K f(gkh) dk = f(g)$ for all $g, h \in G$, or, equivalently, f is annihilated by the Laplace-Beltrami operator), it is easy to see that it is μ -harmonic. Naturally the following question arises :

(A) Under what conditions on μ , μ -harmonic functions are harmonic functions only?

In [6, Theorem 5, p. 370] Furstenberg answers the question above in positive, when μ is absolutely continuous K -invariant probability measure on G/K . In [3] using Winner-Tauberian Theorem, the authors proved the following result for the disc algebra $\mathbb{D} = \text{SL}_2(\mathbb{R})/\text{SO}(2)$. Let Σ denote the usual maximal ideal space $\{z \in \mathbb{C} : 0 \leq \Re z \leq 1\}$.

Theorem 8.1. *Let μ be a $\text{SO}(2)$ -invariant complex measure on \mathbb{D} such that $\mu(\mathbb{D}) = 1$, $\mu(\{0\}) \neq 1$, $\hat{\mu}(\lambda) \neq 1$ for all $\lambda \in \Sigma \setminus \{0, 1\}$, and*

$$\limsup_{x \rightarrow 0^+} x \log |1 - \hat{\mu}(x)| = 0.$$

Then every μ -harmonic functions are essentially the harmonic ones.

Note that the theorem above includes the complex measure too unlike the Furstenberg theorem where the measure is essentially positive. They have also proved that any probability measure μ with $\mu(\{0\}) \neq 1$ satisfies all the conditions of the above theorem. Hence for $\text{SL}(2, \mathbb{R})$ their result contains the Furstenberg Theorem as a particular case.

Since, in this paper, we have obtained the similar Winner-Tauberian Theorem for general hypergeometric transforms (which include all real rank one cases), it is natural to expect that the theorem above holds true for hypergeometric cases. The notion of ' μ -harmonic' does not make sense in general unless the pair (α, β) arises from a geometric case. But this difficulty can be overcome by the following observation. If G is of rank one symmetric space, then writing the Cartan decomposition $G = KAK$, we can identify K -biinvariant functions on G with even functions on $A = \mathbb{R}$. Therefore, taking an average over K , we can write the problem **(A)** in the following equivalent form ((see the proof of [1, Theorem 3.1])):

(B) Let μ be an even complex measure on \mathbb{R} such that $\mu(\mathbb{R}) = 1$. Then under what condition on μ , the only even bounded solutions (on \mathbb{R}) of the equation $f * \mu = f$ are the constant functions. Here, the convolution $*$ is defined by (2.11).

Now we are in position to state the analogues of Theorem 8.1 for our hypergeometric cases. Before stating the theorem, we point out that the choice of maximal ideal space is horizontal strip in our case, where as their is vertical.

Theorem 8.2. *Let μ be an even complex measure on \mathbb{R} such that $\mu(\mathbb{R}) = 1$, $\mu(\{0\}) \neq 1$, $\widehat{\mu}(\lambda) \neq 1$ for all $\lambda \in S_1 \setminus \{\pm i\rho\}$, and*

$$\limsup_{x \rightarrow \rho^-} (\rho - x) \log |1 - \widehat{\mu}(ix)| = 0. \quad (8.1)$$

*Then the only even bounded solutions of the equation $f * \mu = f$ are the constant functions.*

Proof. If f is constant function then, it is easy to see that, $\tau_s f(t) = f(t)$ for all $s, t \in \mathbb{R}$. Therefore

$$f * \mu(t) = f(t) \int_{\mathbb{R}} d\mu(s) = f(t), \quad t \in \mathbb{R}.$$

Conversely, let f be an even bounded function on \mathbb{R} such that $f * \mu = f$. We need to show that f is a constant function. The proof is essentially same as that of the Theorem 8.1. Let I be the closed ideal in $L_0^1(\mathbb{R}, \Delta)_e$ generated by $\mathfrak{S} = (\mu - \delta) * L^1(\mathbb{R}, \Delta)_e$. We shall show that \mathfrak{S} satisfies all the conditions of Theorem 1.1(2). Since $\widehat{\mu}(\lambda) \neq 1$ for all $\lambda \in S_1 \setminus \{\pm i\rho\}$, the common zero set of Fourier-Jacobi transforms of the elements in \mathfrak{S} is $\{\pm i\rho\}$. Also we have, $\widehat{\mu}(t) \rightarrow \mu(\{0\})$ as $t \rightarrow \infty$. But it is given that $\mu(\{0\}) \neq 1$. Therefore it follows that \mathfrak{S} contains an g such that $\delta_\infty^+(g) = 0$. Also (8.1) implies that \mathfrak{S} contains an element h such that $\delta_{i\rho}(h) = 0$. Hence, by Theorem 1.1 (2), we can conclude that $I = L_0^1(\mathbb{R}, \Delta)_e$. Since $f * \mu = f$, clearly, $f * \mathfrak{S} = 0$, and hence $f * L_0^1(\mathbb{R}, \Delta)_e = 0$ which implies that f is a constant function. \square

Corollary 8.3. *Let μ be an even probability measure such that $\mu(\{0\}) \neq 1$. Then the only even bounded solutions of the equation $f * \mu = f$ are the constant functions.*

Again the proof of the above corollary is same as that of [3, Corollary 7.2], once we have the following two lemmas. We shall make use the following derivation property of the hypergeometric function (see [11, p.241, eqn. (9.2.2)]):

$$\frac{d}{dz} {}_2F_1(a, b; c; z) = \frac{ab}{c} {}_2F_1(a+1, b+1; c+1; z), \quad z \in \mathbb{C} \setminus [1, \infty].$$

Helgason-Johnson's theorem states that $|\phi_\lambda| \leq 1$ if and only if $\lambda \in S_1$. We have the following:

Lemma 8.4. *$|\phi_\lambda(t)| < 1$ for all $t > 0$ if and only if $\lambda \in S_1 \setminus \{\pm i\rho\}$.*

Proof. Case 1 : $\lambda \in S_1^0$. Then for $t > 0$

$$\begin{aligned} |\phi_\lambda(t)| \leq \phi_{i\Im\lambda}(t) &= {}_2F_1\left(\frac{\rho + \Im\lambda}{2}, \frac{\rho - \Im\lambda}{2}; \alpha + 1; -\sinh^2 t\right) \\ &= \frac{\Gamma(\alpha + 1)}{\Gamma\left(\frac{\rho + \Im\lambda}{2}\right) \Gamma\left(\frac{\rho - \Im\lambda}{2}\right)} \int_0^1 s^{\frac{\rho - \Im\lambda}{2} - 1} (1 - s)^{\frac{\alpha - \beta + 1 + \Im\lambda}{2} - 1} (1 + s \sinh^2 t)^{-\frac{\rho + \Im\lambda}{2}} ds \\ &< \frac{\Gamma(\alpha + 1)}{\Gamma\left(\frac{\rho + \Im\lambda}{2}\right) \Gamma\left(\frac{\rho - \Im\lambda}{2}\right)} \int_0^1 s^{\frac{\rho - \Im\lambda}{2} - 1} (1 - s)^{\frac{\alpha - \beta + 1 + \Im\lambda}{2} - 1} ds \\ &= \phi_{i\Im\lambda}(0) = 1. \end{aligned}$$

Case 2 : $\lambda = a + i\rho$, $a \neq 0$. Recall the function $G_\lambda^{(\alpha, \beta)}$ from preliminaries.

$$G_\lambda^{(\alpha, \beta)}(t) = \phi_\lambda^{(\alpha, \beta)}(t) + \frac{\rho + i\lambda}{4(\alpha + 1)} (\sinh 2t) \phi_\lambda^{(\alpha+1, \beta+1)}(t), \quad t \in \mathbb{R}.$$

Using the derivation formula of hypergeometric function, we can evaluate $\frac{d^2}{dt^2} \left| G_{a+i\rho}^{(\alpha,\beta)} \right|^2(0)$ to be equal to $-\frac{a^2(2\alpha+1)}{2(\alpha+1)^2}$ which is non-zero. Therefore, $\left| G_{a+i\rho}^{(\alpha,\beta)} \right|^2$ being analytic it can not be identically 1. By [19, Proposition 3.1], $|G_{a+i\rho}^{(\alpha,\beta)}(t)| \leq G_{i\rho}(t) = 1$ for all $t \in \mathbb{R}$. Since $|G_{a+i\rho}^{(\alpha,\beta)}(0)|^2 = 1$, we can have $\epsilon > 0$ such that $|G_{a+i\rho}^{(\alpha,\beta)}(t)|^2 < 1$ for all non-zero t with $|t| \leq \epsilon$. But, in the proof of the [19, Proposition 3.1] it is shown that

$$t \rightarrow \max \left\{ |G_{a+i\rho}^{(\alpha,\beta)}(t)|^2, |G_{a+i\rho}^{(\alpha,\beta)}(-t)|^2 \right\}$$

is a decreasing function of $t \geq 0$. So it follows that $|G_{a+i\rho}^{(\alpha,\beta)}(t)|$ is strictly less than 1 for all $t \neq 0$. Since

$$\phi_\lambda^{(\alpha,\beta)}(t) = \frac{1}{2} \left[G_\lambda^{(\alpha,\beta)}(t) + G_\lambda^{(\alpha,\beta)}(-t) \right], \quad t \in \mathbb{R},$$

the proof follows. □

Lemma 8.5. *Let $t > 0$. Then $\frac{d}{dx} \big|_{x=\rho} [\phi_{ix}(t)] > 0$.*

Proof. Define the function g on \mathbb{R} by

$$g(t) = \frac{d}{dx} \bigg|_{x=\rho} [\phi_{ix}(t)].$$

Then $g(0) = 0$ and

$$\begin{aligned} g'(t) &= \frac{d}{dx} \bigg|_{x=\rho} \left(\frac{d}{dt} [\phi_{ix}(t)] \right) \\ &= \frac{d}{dx} \bigg|_{x=\rho} \left[\frac{(\frac{\rho+x}{2})(\frac{\rho-x}{2})}{\alpha+1} {}_2F_1 \left(\frac{\rho+x}{2} + 1, \frac{\rho-x}{2} + 1; \alpha+2; -\sinh^2 t \right) (-\sinh 2t) \right] \\ &= \frac{\rho \sinh t}{2(\alpha+1)} {}_2F_1(\rho+1, 1; \alpha+2; -\sinh^2 t) \\ &= \frac{\sinh t}{2\Gamma(\rho)} \int_0^\infty (1-s)^\alpha (1+s \sinh^2 t)^{-\rho-1} ds \end{aligned}$$

which is strictly positive whenever $t > 0$. Therefore g is strictly increasing function on $[0, \infty)$ and hence $g(t) > 0$ for all $t > 0$. □

Proof. of Corollary 8.3 : We only need to show that the measure μ satisfies all the conditions of Theorem 8.2. Since μ is a probability measure $\mu(\mathbb{R}) = 1$; $\mu(\{0\}) \neq 1$ is given, in fact, $\mu(\{0\}) < 1$ since μ is positive. If $\lambda \in S_1 \setminus \{\pm i\rho\}$, then by Lemma 8.4,

$$|\widehat{\mu}(\lambda)| \leq \int_{\mathbb{R}} |\phi_\lambda^{(\alpha,\beta)}(t)| d\mu(t) < \int_{\mathbb{R}} d\mu(t) = 1.$$

Therefore we only left to show that $\limsup_{x \rightarrow \rho-} (\rho - x) \log |1 - \widehat{\mu}(ix)| = 0$. For $t \geq 0$, let

$$L(t) = \frac{d}{dx} \bigg|_{x=\rho} [\phi_{ix}^{(\alpha,\beta)}(t)].$$

Since, by Lemma 8.5, $L(t)$ is strictly positive for all $t > 0$ and, by the given condition, μ is not concentrated at 0 there exist $b > a > 0$ such that

$$\int_a^b L(t) d\mu(t) > 0.$$

Fix $0 < \epsilon < \rho$. From the Taylor series expansion (upto second order) of the function $x \rightarrow \phi_{ix}(t)$ at the point $x = \rho$, it follows that

$$1 - \phi_{ix}(t) \asymp (\rho - x)L(t), \text{ for all } x \in [\rho - \epsilon, \rho], t \in [a, b].$$

Therefore, for all $x \in [\rho - \epsilon, \rho]$,

$$1 - \widehat{\mu}(ix) = 2 \int_0^\infty (1 - \phi_{ix}(t)) d\mu(t) \geq 2 \int_a^b (1 - \phi_{ix}(t)) d\mu(t) \geq 2C(\rho - x) \int_a^b L(t) d\mu(t)$$

where C is a positive constant which depends only on ϵ, a, b . Therefore it follows that

$$\limsup_{x \rightarrow \rho^-} (\rho - x) \log |1 - \widehat{\mu}(ix)| = 0.$$

as desired. □

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